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Common origin of all paradoxes and incompleteness in logic-- Introduction of the BA-3 indefinite state B in the Boolean Vector-Matrix Formulation of Logic for the truth value of a statement which is neither provable to be true nor false

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Abstract

We have pursued the consequences of the triple isomorphism between the set of subsets S_a, S_b, S_c etc., of a universal set S, the logic of the properties A, B, C etc., of the members of these subsets and the Boolean numbers 1, 0 for the elements a, b, c etc., of a Boolean algebra (BA-1) of genus 1, along with the 1—1 correspondences between the operators $\cup$, $\cap$, $\sim$ of set theory, $\vee$, $\wedge$, $\neg$ of propositional calculus and Boolean sum ($\oplus$), Boolean product ($\otimes$) and Boolean complement (^{c}) of BA-1. The reversel of a relation of the type $S_a \cap S_b = S_c \longleftrightarrow A \wedge B = C \longleftrightarrow a \otimes b = c$, expressed in the form "Given the truth value T or F of A and C, what is the truth value of B?", leads to the possibility of there being other states of truth of the nature $T \vee F$ (= D, tautology) and $T \wedge F$ (= X, contradiction), which are not describable by BA-1, but only by 2-vectors of the type $\underline{\underline{a}} = (a_\alpha \quad a_\beta)$ is Boolean algebra BA-2 (having (1 0) for T, (0 1) for F, (1 1) for D and (0 0) for X), together with 2x2 matrices for logical connectives. The whole of propositional calculus (PC) has thus an isomorphic representation in this BA-2 algebra, which has been named SNS (standing for Syad Nyaya System, Syad = may be, Nyaya = Logic, in Sanskrit).

The Boolean operations $\oplus$ and $\otimes$ are applicable to n-vectors in the form $\underset{\sim}{\underline{a}} \text{\textcircled{$\oplus$}} \underset{\sim}{\underline{b}} = \underset{\sim}{\underline{c}} \longleftrightarrow a(\underline{i}) \oplus b(\underline{i}) = c(\underline{i})$ ($\underline{i}$ = 1 to $\underline{n}$) and similarly for $\text{\textcircled{$\otimes$}}$, and ^{c} applied to n-vectors. Then, in addition to matrix operators which correspond to the logical operators $\wedge$, $\vee$, $\neg$ etc, we also have Boolean operators for the 'superposition' of information for the same logical term from two independent sources in the form $\underset{\sim}{\underline{a}}_1 \text{\textcircled{$\oplus$}} \underset{\sim}{\underline{a}}_2 = \underset{\sim}{\underline{a}}$

and $\underset{\approx}{a}_1 \otimes \underset{\approx}{a}_2 = \underset{\approx}{a}$. These correspond to checks for presence and absence of contradiction and for the occurrence of tautology in classical logic, as well as for drawing the effective truth value of $\underset{\approx}{a}$ from those of $\underset{\approx}{a}_1$ and $\underset{\approx}{a}_2$, but are neatly representable by Boolean symbols and operators in BA-2 for PC and BA-n for multivalued logic (MVL).

The really interesting consequence that comes out of this approach is from the reversal of such Boolean operations in BA-2, when the resulting truth values are not expressible even in BA-2, but only in BA-3, eg. of the type $\underset{\sim}{a}$ = 'D but not T', 'D but not F', 'D but not T and not F' by 3-element vectors $\underset{\sim}{a} = (a_\gamma \ a_\delta \ a_\epsilon)$. The three basic vectors of BA-3 are (1 0 0) for 'always T' (A T), (0 0 1) for 'always F' (AF) and (0 1 0) for 'D but not T and not F' (B). This third state 'B' of truth, representing the logical property of 'exists, but not definitely true and not definitely false' is the 'excluded middle' of classical logic. It has been included by Brouwer in his Intuitionistic Logic, and since the Boolean algebraic state (0 1 0) has all the properties of this Brouwer state, we have given it the symbol B. In fact, the BA-3 algebra can also represent the various quantifiers of predicate logic, with $(1\ 0\ 0) \longleftrightarrow (\forall x)(ax)$, $(0\ 0\ 1) \longleftrightarrow (\forall x)(\neg ax)$, and $(0\ 1\ 0) \longleftrightarrow (\sum x)(ax) = (\exists x)(ax) \wedge (\exists x)(\neg ax)$. Hence, BA-3 has $(2^3 - 1) = 7$ non-zero elements plus one corresponding to the null set $(\phi x)(ax)$. In fact, the seven states of truth were listed by the ancient Indian Jaina philosophers as '<u>saptabhangi</u>' (= seven-fold, in Sanskrit) with the name '<u>avaktavya</u>' (inexpressible) for the state (0 1 0)— hence the name SBL (Saptabhangi Logic) for logic with BA-3 states.

With this background, which is mainly presented in the appendix, the result in the main body of the paper become apparent. It is shown that a variety of paradoxes and antimonies that are present in the literature of logic can all be attributed to the existence in all of them of a common relation of the type $\underline{\underline{a}} \Longleftrightarrow \neg \underline{\underline{a}}$ for a statement $\underline{\underline{a}}$, where $\neg$ corresponds to the interchange of truth and falsity. This equation is usually assumed to have no solution in classical logic— hence the paradox , but it has two solutions in SNS, namely $(T \bigvee F)$ and $(T \bigwedge F)$, corresponding to universal doubt (D) and contradiction (X), both of which are 'singularities' in logic, However, if BA-3 truth values are included, the state (1 1 1) corresponding to D disappears, and is replaced by the state B (0 1 0) of SBL, which corresponds to the result that the statement $\underline{\underline{a}}$ "exists, but is not provable to be definitely true, or definitely false'. The other solution X continues as XX (0 0 0) in SBL, corresponding to 'contradiction'. So also, Godel's proof of incompleteness of predicate logic does not necessarily lead to any defect in the axioms, but has a logical representation in SBL in this state B. Hence, the construction of a statement $\underset{\sim}{G}$ which is "true, but not derivable to be true or false" only indicates that $\underset{\sim}{G}$ has the permissible state (0 1 0) of BA-3, and does not react on the axioms of the field of knowledge in which the statement is proposed. It only indicates the need for the expansion of the <u>semantics</u> of the field of knowledge by incorporating the consequence of the 'singular' state B arising for $t(\underset{\sim}{G})$, which is impossible in CL or SNS. An analysis of some 9 paradoxes and incompletenesses are given including also the belief in ancient Indian philosophy that there is a Supreme Being encompassing the whole of existence, but for which the only definition is by the paradoxical statement, "We know It when we realise that It can never be known fully".

Common origin of all paradoxes and incompleteness in logic — Introduction of the BA-3 indefinite state B in the Boolean Vector-Matrix Formulation of Logic for the truth value of a statement which is neither provable to be true nor false

G.N. Ramachandran

1. Introduction

This report contains a unified treatment on the basis of BVMF[1] of various paradoxes like the Cretan Liar Paradox, Richard's Paradox, Russell's Paradox regarding sets, the paradox of infinity, as well as the existence of incompleteness, as was shown by Godel, for predicate logic, in 1931. All these could be attributed to the fact that logic has not only two classical truth values "True" (T) and "False" (F) of BA-1, but necessarily requires two other truth values — "Doubtful" (D = 'T or F') and "Impossible" (X = 'T and F') which are present in BA-2 algebra. The last of these is a contradiction, but the third one is a possible truth value for some statements which have 'singular' properties — particularly of the type $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$. This is because $\neg T = F$ and $\neg F = T$, so that $\neg(T \oplus F)$ is once again $(T \oplus F)$, and the state D is a non-contradictory solution of this equation. It is shown that in all the above paradoxes, as well as in Godel's proof and related incompletenesses, the above equation $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$ is the essential operative condition for the singular nature of the term $\underline{\underline{a}}$.

One further modification of this arises from the fact that the pure states T and F do not satisfy the equation $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$ while the mixed state D = 'T or F' satisfies it. Therefore, the exact solution of $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$, other than "contradiction", is B = "D, but not T and not F" and not D itself. In BVMF, this is a basic state in BA-3, which has seven possible truth values and an eighth impossible one, or "contradiction", built up from the three basic states T, B and F. (see [R2], [R3], [R4], [R5]).

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[1] BVMF stands for "Boolean Vector-Matrix Formulation" of Logic developed by the author [R1, R2 R3]. A glossary is given at the end of the report of the symbols, of terms and other abbreviations in BVMF that are used in this report.

It is interesting that this truth value is identical with the "third state" that is postulated in Brouwer's intuitionistic logic which does not obey the Law of Excluded Middle. Therefore, the letter B, which is the first letter in the name Brouwer, has been adopted as the symbol for this truth value. It corresponds to the fact that the statement under consideration cannot be proved to be either true or false. Such a conclusion need not, however, mean that the statement should be considered invalid, or that the axioms of the relevant field of knowledge and/or of logic are incomplete. This is discussed in detail in what follows, employing BVMF formulae and techniques.[2] It is shown in particular that Godel's proof of incompleteness of the axioms of arithmetic is not valid and that his results pointing to "incompleteness" are consistent with the BA-3 algebra of quantifier calculus. Thus, the concept that Godel's proof leads to incompleteness of the axioms of arithmetic is found to be not necessarily true, and that we can only say that the statement G (G1, NN-1) contained in [13] Section)) has a truth value B which is a logically permissible feature. The paradoxes are also all shown to have a non-contradictory solution in the permissible logical state B.

The possibility of perpetual doubt for some logical statements has examples not only in logic and mathematics ; it has even been noted in philosophy, particularly of the eclectic type in ancient Indian (Hindu and Jaina) philosophical literature. In fact, the Jainas had seen the need for the four truth values T, F, D, X (of BA-2 in BVMF) and even more surprisingly, the seven states of truth of BA-3, which they had named Saptha-Bhangi (seven-membered system) and had given a name 'avaktavya' (indescribable) for the indefinite state B [H1] . In Hindu philosophy of the Upanishads, even the Ultimate Reality (Brahman) is found to be capable of definition and description only in terms of such apparently paradoxical statements as — "It is not known to those who think they know It ; but It is known to those who know they cannot know it". (See [R9] in particular for a detailed discussion of this, which is briefly presented in Section 3(ix)).

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[2] A brief indication of BVMF formulae and techniques is given in the Appendix A. The appendix A and the glossary together should be sufficient for reading this report from the BVMF point of view.

2. Definition of the third indefinite state B and its consequences

A number of paradoxes (antimonies) has appeared in the literature of logic right from the classical Greek period going through the ages and more precisely stated in the treatments in modern texts on mathematical logic. A representative (but not exhaustive) list of these is given in the next section and the nature of the resolution of each of them is indicated therein. It turns out that all such "singularities" in logic have one feature in common, namely the simultaneous occurrence of the two equations (1a) and (1b) for some statement $\underline{\underline{a}}$:

$$\underline{\underline{a}} \Longrightarrow \neg\, \underline{\underline{a}} \quad \text{and} \quad \neg\, \underline{\underline{a}} \Longrightarrow \underline{\underline{a}} \tag{1a, b}$$

or put in words,

"If $\underline{\underline{a}}$ is true then $\underline{\underline{a}}$ is not true, and
if $\underline{\underline{a}}$ is not true then $\underline{\underline{a}}$ is true" (2)

In the form of Eq.(1), the two implications can be combined to lead to the equivalence in (3a) and (3b)

$$(\underline{\underline{a}} \Longleftrightarrow \neg\, \underline{\underline{a}}) \equiv (\neg\, \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}) \tag{3a,b}$$

The question therefore is one of finding a solution or solutions for $\underline{\underline{a}}$ (under the conditions of the problem) which satisfy Eq.(3) or Eqs. (1a,b).

This is best done in BVMF for propositional calculus in the SNS[3] formulation [R1] employing Boolean algebra of genus 2 (BA-2). Denoting

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[3] SNS stands for "Syad Nyaya System" (Syad = May be, Nyaya = logic, in Sanskrit) and represents BA-2 truth value generated by two basic states T, F. We shall always use SBL for BA-3 truth values (SBL standing for Sapthabhangi (seven-membered system) Logic), generated by three basic states T, B, F.

the vector $\underline{\underline{a}}$ by $\langle \underline{a} |$ and the operator connecting the left and right sides of Eq. (3) by the matrix $|\underline{Z}| = |\underline{N}|$, it has the form

$$\langle \underline{a} | \underline{N} | = \langle \underline{a} | , \quad | \underline{N} | = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \langle \underline{a} | = (a_{\alpha} \quad a_{\beta}) \tag{4}$$

Putting the only two possible values 0 and 1 for the Boolean scalars a_{α} and a_{β} , the only possible solutions of Eq. (4) come out to be

$$a_{\alpha} = 1 , \quad a_{\beta} = 1, \quad \langle \underline{a} | = (1 \quad 1) = D \tag{5a}$$

$$a_{\alpha} = 0 , \quad a_{\beta} = 0, \quad \langle \underline{a} | = (0 \quad 0) = X \tag{5b}$$

Which one of the two solutions (5a) and (5b) is valid for a particular problem and when one, or the other, or both, become possible, depends on the semantics of the argument under consideration which leads to Eq.(1). However, as we shall show below, one or the other or these (or both) gives a resolution of the peculiar situation of what is commonly considered to be a paradoxical one, or one leading to the conclusion that the axioms of some field of knowledge are incomplete or defective.

As has been described in [R1] and [R8], the state D of SNS <u>includes</u> the possibility of being pure T, and pure F, by themselves, but neither of these satisfies Eq.(1) or (3). (It will be noticed that for both T = (1 0) and F = (0 1), the equation $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$ is not valid.) Therefore Eq. (5a) will have to be modified as (5c) to say:

$$\underline{\underline{a}} = \text{"D but not T and not F"} = B \quad \text{(say)} \tag{5c}$$

This is not a state for a truth value in BA-2 and therefore we have to go further to BA-3 which we have shown in [R2], [R3] and [R5] is valid for quantifier calculus. When this calculus is applied to truth values, the basic vectors are

$$T = (1\ 0\ 0), \quad F = (0\ 0\ 1), \quad B = (0\ 1\ 0) \tag{6}$$

for the three orthogonal truth values for 3-valued logic, corresponding to the three quantifier states $\forall$ (For all) $= (1\ 0\ 0)$, $\bar{\Phi}$ (For none) $= (0\ 0\ 1)$, Σ (For some) $= (0\ 1\ 0)$. (The symbol B for the analog Σ in QPL, is after the first letter of the name of Brouwer who first introduced this indefinite state in his Intutionistic Logic [B1] and [B2]).

Since the operator $\underline{\underline{N}}$ only converts T into F and F into T, the corresponding matrix is, in BA-3,

$$|\underset{\sim}{N}| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{7}$$

It will be seen that B remains invariant under its operation to give the equation (8) for (3) in BA-3 :

$$\langle \underset{\sim}{a} | \underset{\sim}{N} | = \langle \underset{\sim}{a} | \quad \text{for} \quad \langle \underset{\sim}{a} | = B = (0\ 1\ 0) \tag{8}$$

In BA-3, the analog of X = (0 0) of SNS, is XX = (0 0 0) and this also satisfies Eq.(8) so that (5a) and (5b) when properly revised take the form,

$$a_\gamma = 0\ , \quad a_\delta = 1, \quad a_\epsilon = 0\ , \langle \underset{\sim}{a} \backslash = (0\ \ 1\ \ 0) = B\ (\Sigma \text{ in QPL}) \tag{9a}$$

$$a_\gamma = 0\ , \quad a_\delta = 0, \quad a_\epsilon = 0\ , \langle \underset{\sim}{a} \backslash = (0\ \ 0\ \ 0) = XX(\phi \text{ in QPL}) \tag{9b}$$

These two solutions are, however, not the only ones for Eq.(8) in BA-3. There are also two other solutions of the equation $\langle \underset{\sim}{a} \backslash \underset{\sim}{N} | = \langle \underset{\sim}{a} |$, namely

$$a_{\gamma} = 1, \quad a_{\delta} = 0, \quad a_{\epsilon} = 1, \langle \underset{\sim}{a} | = (1\ 0\ 1) \ = B^{c} \ (\bigwedge \text{ in QPL}) \qquad (10a)$$

$$a_{\gamma} = 1, \quad a_{\delta} = 1, \quad a_{\epsilon} = 1, \langle \underset{\sim}{a} | = (1\ 1\ 1) \ = DD \ (\Delta \text{ in QPL}) \qquad (10b)$$

However, both these contain the definite states of truth and falsity, namely (1 0 0) and (0 0 1), as possibilities that can occur, which, however, do not satisfy Eq.(1). They are therefore not valid solutions of Eq.(8), and we are left with (9a) and (9b) as the only solutions. These two can be given a simple logical description in terms of 'absolutely true' and 'absolutely false' (AT and AF) and 'possibly true' and'possibly false' (ET and EF) states using the vidya operator $\underset{\sim}{\mathrm{V}}$. [R2, R8]

$$XX = (0\ 0\ 0) \ = \ (1\ 0\ 0) \textcircled{\otimes} (0\ 0\ 1) \ = \ AT \ \underset{\sim}{\mathrm{V}} \ AF$$ = "both absolutely true and absolutely false " (Contradiction) (11a)

$$B \ = (0\ 1\ 0) \ = \ (1\ 1\ 0) \textcircled{\otimes} (0\ 1\ 1) \ = \ ET \ \underset{\sim}{\mathrm{V}} \ EF$$ = "possibly true and possibly false but neither definitely so " (Indefinite) (11b)

The analogs of the states AT, ET, AF, EF in QPL are $\forall$, $\exists$, $\neg\exists$, and $\neg\forall$ respectively.

As will be seen from the above BVMF description of this state B, it is a state that "exists"; but if a term or statement $\underset{\sim}{a}$ is in that state, its truth cannot be demonstrated definitely by deduction since $a_{\gamma} = 0$, and similarly its definite falsity cannot also be proved since $a_{\epsilon} = 0$. This follows from the BVMF formulae [R3]

$$B \ \underset{\sim}{\mathrm{V}} \ AT \ = \ (0\ 1\ 0) \textcircled{\otimes} (1\ 0\ 0) = (0\ 0\ 0)$$ — "B is 'orthogonal' to 'absolutely true' in BA-3 " (12a)

$$B \ \underset{\sim}{\mathrm{V}} \ AF \ = \ (0\ 1\ 0) \textcircled{\otimes} (0\ 0\ 1) = (0\ 0\ 0)$$ — "B is 'orthogonal' to 'absolutely false' in BA-3 " (12b)

Thus we get the strange result that a logical term, or statement, can arise in an argument statable in terms of BA-1 or BA-2 algebra for truth values, but whose truth value is such that neither its absolute truth nor absolute falsity is derivable or provable; but yet the statement exists (or "is true"). It is interesting that this is precisely the way in which Godel's incompleteness theorem for predicate calculus [G1 , NN1] is stated

> "There is an arithmetical formula $\underset{\sim}{G}$ in predicate calculus that is such that neither the truth nor the falsity of $\underset{\sim}{G}$ can be demonstrated although $\underset{\sim}{G}$ is a valid permissible statement in the system". (13)

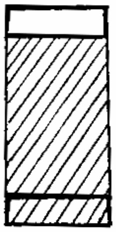
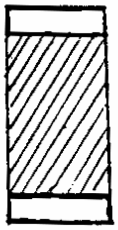
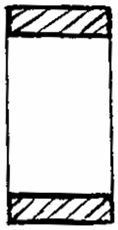
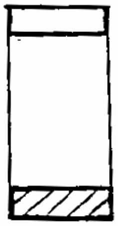
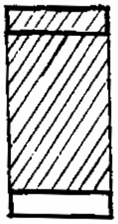
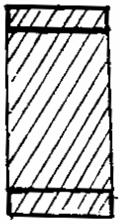
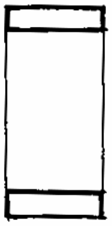
However, as can be readily seen from our formulation given above, the non-decidability of such a statement does not necessarily reflect on the consistency or completeness of the axioms of the system, or field of knowledge, in which the argument is stated, but could possibly be merely a reflection on the very nature of two-valued logic, which demands 3-valued logic (BA-2) and then 7-valued logic (BA-3) even for an argument stated in classical propositional calculus in BA-1 [R8]. Thus it has been shown by us that just as the set of integers which form a ring under the operation of + , - and × require an extension to rational numbers of the type p/q when the multiplication is reversed so to include its inverse operation (division) in the system, in the same way the BA-1 algebra, which is a Boolean ring, necessarily requires BA-2 truth values on reversal of some of the relations obtained in BA-1 (see Appendix A). Further, a similar reversal of some of the relations in BA-2 which is also a Boolean ring will lead to BA-3 truth values. Therefore, BA-3 truth values of the type we have discussed above exist in logic independent of any semantic content or axiomatic nature of the terms involved in the statement which is shown to have the indefinite state $B = (0 \quad 1 \quad 0)$. We shall explain and illustrate this fact further with several examples in what follows. (For a proof of the extension of BA-1 to BA-2 and BA-3 by reversal of relations, see Appendix A and the earlier references given therein.)

What is more interesting is that the third state B which is one of the three generators of BA-3, namely $(0 \quad 1 \quad 0)$, is identical with the third truth value for the excluded middle of logic which was postulated by Brouwer [B1] in his Intuitionistic Logic, where he indicates that there could be acceptable statements (theorems) of logic which are such that they can neither be shown to be definitely true nor definitely false, but simply have this third truth value which is neither T nor F but is midway between. As an example, Brouwer indicates a possibility that there may be no proof available for the famous Fermat's theorem — namely "The equation $x^n + y^n = z^n$ has no integer solutions x, y, z for all $n > 2$ "—either in the positive or negative sense — and that the theorem may be unprovable to be either definitely true or definitely false, ie., its truth value is the B state discussed above.

Thus, we find that BA-3 algebra containing particularly the third indefinite state (0 1 0) other than the two definite states (1 0 0) and (0 0 1) indicating absolute truth and absolute falsehood has several applications in logic. This is indicated by the comparative Table 1 below in which the notation and essential properties of BA-3 algebra are listed. Table 2 contains some of the more important examples of the use of the indefinite state B. (See Discussion section for further details.)

In fact, the "indefinite" state B not only finds application in indicating incompleteness of the inputs for an argument or the resolution of paradoxes, but it has great application in the development of all fields of knowledge <u>via</u> a number of stages at each of which the occurrence of state B directly points the way in which the indefiniteness can be resolved by including an additional hypothesis or axiom in the input. We shall illustrate this by two examples from mathematics and also indicate one example from ancient Hindu philosophy where the very concept or picture that is sought to be conveyed cannot be done in any way other than by means of a paradox leading to the concept itself being in the state B.

Table 1. The 7 + 1 elements of BA-3 algebra and the corresponding entities namely quantifiers in QPL and truth values in SBL.

Graphical illustration	Quantifier state in QPL	BA-3 symbol & notation	Truth value in SBL	Graphical illustration	Quantifier state in QPL	BA-3 symbol & notation	Truth valu in SBL
	For all $\forall$	(1 0 0) q(1)	Always true AT		Not for all $\Lambda = (\forall^c)$	(0 1 1) q(2)	Exists falsehood EF
	For some Σ	(0 1 0) q(3)	Indefinite B		All or none $\Theta = (\Sigma^c)$	(1 0 1) q(4)	Definite A
	For none Φ	(0 0 1) q(5)	Always false AF		There exists $\exists\ (= \Phi^c)$	(0 1 0) q(6)	Exists truth ET
	Full set Δ	(1 1 1) q(7)	Tautology D D		Null set $\phi\ (= \Delta^c)$	(0 0 0) q(8)	Contradiction X X

In terms of the standard quantifiers that are used in predicate logic, namely $\forall$ and $\exists$ the other six quantifiers contained in the full set of eight BA-3 elements are
$\Lambda = \neg\forall$, $\Phi = \neg\exists$, $\Sigma = \exists \otimes \exists\neg$, $\Theta = \forall \oplus \forall\neg$, $\phi = \forall \otimes \neg\forall = \exists \otimes$
$\Delta = \exists \oplus \neg\exists = \forall \oplus \neg\forall$

Table 2. Examples of the occurrence of the indefinite state B in logic and BVMF*

Application	Quantifier state or symbol	Where employed
BA-3 algebra	(0 1 0)	One of the three generator states (1 0 0), (0 1 0), (0 0 1)
Quantifier Calculus (QPL)	Σ (For some)	$\Sigma \equiv \exists \veebar \neg \forall$, standing for "For some" $\equiv$ "There exists, but not for all"
Saptabhangi Logic (SBL)	B (Indefinite)	"Exists, but not always true (and not always false)"
Brouwer's Intuitionistic Logic	B (Excluded Middle)	The third state of truth, that forms the "excluded middle" of classical logic.
Godel's Theorem	'G'	Example, in number theory, of a statement that is neither provable nor disprovable, but is still true and valid.
Paradoxes	B	Leading to their explanation, and the incorporation of statements that satisfy the equation $\neg a \Longleftrightarrow a$ into the domain of acceptable logic.

* Only those considered in this report are included. Obviously, there are many more examples and applications of this type as mentioned in the text.

All these will be presented in short detail in the succeeding sections. Fuller reports of these consequences are expected to be prepared in due course.

3. Discussion of Paradoxes and Incompletenesses[4]

In this section, we shall give a brief account of a number of paradoxes (including incompleteness of the Godel's type) and indicate briefly how the resolution of the paradoxical result $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$ can be obtained to be one of the three types (14a,b,c)

1) $\underline{\underline{a}}$ being either X X or B (14a)
2) $\underline{\underline{a}}$ being necessarily X X , (14b)
3) $\underline{\underline{a}}$ being necessarily B (14c)

which will cover all the range of examples of this type of singularity that is available in the literature. Anticipating what will be shown later, the first case (14a) occurs with paradoxes which are essentially created by a statement which refers to itself in the negative sense, and such a statement can be considered to be contradictory (X X) or perpetually indefinite (B) depending upon the problem. The second case (14b) occurs when the indefinite state is not permissible. The third case, on the other hand, occurs surprisingly in a large number of examples and we will show that the history of mathematics contains a very large number of examples which can be classified under this category — e.g the paradox of infinity which makes infinity both greater and smaller than itself so that no definite conclusion can be obtained about one or the other being valid. We show also that Godel's proof should also be categorised under this class and not indicating any inconsistency in the axioms of arithmetic.

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[4] A fairly complete account of such paradoxes and incompleteness theorems is contained in [EB1]

(i) Cretan Liar Paradox

This paradox is one of the earliest in Greek philosophy, and it perhaps indicates in a nutshell the essence of our point of view. The paradox is as follows:

> A person (Cretan Liar) says "What I say is not true". Is it (what he says) really true or false?

It is obvious that if what he says ($\underline{\underline{a}}$)is T then the contents of the statement indicates that $\underline{\underline{a}}$ is F . Then the contents indicate that this falsity is invalid so that $\underline{\underline{a}}$ is T. In other words, we obtain the two implicational relations $\underline{\underline{a}} \Longrightarrow \neg \underline{\underline{a}}$ and $\neg \underline{\underline{a}} \Longrightarrow \underline{\underline{a}}$ identical with our Eqs. (1a,b). The two together are equivalent to $\neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}}$ in Eq.(3).

As we have indicated in the previous sections, there are two possible solutions to the problem, namely corresponding to the BVMF states X standing for contradiction and B standing for indefiniteness. In the former case, we could conclude that "there is no such person as Cretan Liar" corresponding to $\underline{\underline{a}}$ being contradictory. On the other hand, if $\underline{\underline{a}}$ corresponds to the state B, then we may say "it is not possible to say whether what the Cretan Liar says is T or F." In either case, it will be noted that nothing is to be said about the axioms of logic or any system of knowledge. Such initial information that are fed into this simple argument are perfectly sound, and the conclusions we arrive at are valid according to the rules of logic. Thus, if we apply the added framework of BA-3 algebra of SBL for truth values, we conclude that there is a complete answer to the paradox in permissible logical statements mentioned above. Which of the conclusions is chosen for further application depends upon the semantics connected with the conditions of the problem and do not involve any considerations of logic.

It is commonly believed that the Cretan Liar statement is inadmissible because the statement comments on itself and as such inadmissible. We shall now consider the next example of the double statement paradox in which the conclusion is the same as in the Cretan Liar paradox, but a statement is related to itself only <u>via</u> a circuit of connected statements and not directly to itself.

<u>(ii) Double Statement Paradox</u>

Briefly stated, it consists of the following two statements

A says that $\underline{\underline{a}}$: 'What B says' ($\underline{\underline{b}}$) is false (15a)

B says that $\underline{\underline{b}}$: 'What A says' ($\underline{\underline{a}}$) is true (15b)

It is then seen that we can combine the two statements in two different ways and arrive at such equations as in (16a,b):

$$\underline{\underline{a}} \Longleftrightarrow \neg \underline{\underline{b}}\,,\quad \underline{\underline{b}} \Longleftrightarrow \underline{\underline{a}} \quad \longmapsto \quad \neg \underline{\underline{a}} \Longleftrightarrow \underline{\underline{a}} \tag{16a}$$

$$\underline{\underline{b}} \Longleftrightarrow \underline{\underline{a}}\,,\quad \underline{\underline{a}} \Longleftrightarrow \neg \underline{\underline{b}} \quad \longmapsto \quad \neg \underline{\underline{b}} \Longleftrightarrow \underline{\underline{b}} \tag{16b}$$

In this case, it is seen that both $\underline{\underline{a}}$ and $\underline{\underline{b}}$ satisfy Eq.(1) and the same conclusion as for the Cretan Liar hold for both the persons A and B and to what they say. It must, however, be remembered that it is sufficient for one of them to be in the indefinite state B for both the Eqs. (15a, b) to be made non-paradoxical.

Clearly, a set of implications or equivalences connecting $\underline{\underline{a}}$ with $\underline{\underline{b1}}$, $\underline{\underline{b2}}$, $\underline{\underline{b3}}$, ... , $\underline{\underline{bn}}$ and back to $\underline{\underline{a}}$ itself can occur in any argument effectively leading to $\underline{\underline{a}} \Longrightarrow \neg \underline{\underline{a}}$ or $\neg \underline{\underline{a}} \Longrightarrow \underline{\underline{a}}$. In such cases, the conclusion arrived at in standard logic is that, respectively, $\underline{\underline{a}}$ or $\neg \underline{\underline{a}}$ is not possible. If both occur , then $\underline{\underline{a}} \Longleftrightarrow \neg \underline{\underline{a}}$ (Eq.3). Since this commonly occurring situation of $\underline{\underline{a}}$ referring to itself effectively leads to a situation similar to the Cretan Liar Paradox, it can be concluded that a statement referring to itself is not the contributing factor for the occurrence of the paradox and that the paradox does not exist if we accept BA-3 algebra for truth values and admit that the indefinite state B is valid and permissible.

(iii) Barber of Seville problem

This famous paradox from medeaval times says:

> "The barber of Seville shaves all those and only those who do not shave themselves. Does he shave himself or not?"

If the statement "He shaves himself " is designated by the symbol $\underline{\underline{a}}$, then it is obviously seen that they lead to the same two equations $\underline{\underline{a}} \Longrightarrow \neg \underline{\underline{a}}$ and $\neg \underline{\underline{a}} \Longrightarrow \underline{\underline{a}}$ as in Eq.(1), and therefore just as in the Cretan Liar Paradox, we can conclude that one of the following two possibilities is valid.

Either (a) that the statement $\underline{\underline{a}}$ is in the state X X, and is an impossible or contradictory statement in logic, so that we can conclude that the barber of Seville is not existent,

or (b) that the statement $\underline{\underline{a}}$ has the indefinite state B and therefore it is perpetually doubtful whether the barber of Seville shaves himself or not and no definite information on this can ever be obtained from this line of enquiry.

Clearly a non-contradictory answer to the paradox can only be given in the state B but not in AT or AF.

It is to be noticed that all the three examples mentioned above are in propositional calculus and do not involve any quantifier, and they have no solution in classical logic, or in SNS logic, with T and F as the basic truth values, for the only solution permissible under this is X which is a contradiction. However, even without the use of quantifiers, the truth value itself can have an admissible state B having BA-3 algebra and it is to be noted that there is no kick-back from this conclusion to the axioms or the inputs to each of the paradoxes. We shall now consider another famous paradox connected with the notion of an infinite integer (∞) in which case the solution XX seems to be not possible and the only solution is B.

(iv) Cantor's paradox

This occurs in several paradoxes connected with infinity, all of which stem from the generic equation: (RUS 1)

$$\infty = \infty + 1 \tag{17}$$

which may be taken to explain the difference in property of this number ∞ from any finite number. This difference is most clearly indicated by Cantor's paradox of infinite sets which asks the question: "Does the set of all sets contain itself or not?" Once again, we get the answer in the form of Eq.(1a,b), for if we take $\underline{\underline{a}}$ to stand for "The set of all sets contains itself", then since it is the set of all sets, it cannot contain itself ($\underline{\underline{a}} \Longrightarrow \neg \underline{\underline{a}}$), and if it does not contain itself, then it is not the set of all sets, so that $\neg \underline{\underline{a}} \Longrightarrow \underline{\underline{a}}$, resulting in $\underline{\underline{a}} \Longleftrightarrow \neg \underline{\underline{a}}$, which is Eq. (3).

In this case, the solution X is not possible since we do know from mathematical analysis that infinite sets exist (in concept, if not in reality). Therefore, the solution $\underset{=}{a} = X$ or $\underset{\sim}{a} = X\ X$ is to be rejected from semantic grounds (not logical). On the other hand, the solution $\underset{\sim}{a} = B$ is perfectly permissible and it is in fact very reasonable. It means that it is impossible to say for certain whether the set of all sets contains itself or not. It is a question that can only be answered in the state B i.e "may be so or may not be so". It turns out that this indefiniteness gives a valid definition of the infinity (of integers), because both the statements — "It is greater than itself" or "It is not greater than itself" are unprovable, and this can be considered to be a definition of infinity. The generic equation (17) requires this, and may be taken to be required in order to take care of the paradox.

The point we are making is that the analog of the antimony in Eq.(1) in the case of Cantor's paradox, does not reflect in any way on the axioms of logic or number theory, but in fact points the way to an extension of the theory of finite integers to include ∞. How this extension is made is not relevant to the resolution of the paradox, but clearly the paradox indicates no inconsistency in the axioms. As is well-known, Cantor developed his theory of cardinal numbers connected with infinite sets to deal with this difficulty which effectively puts ∞ and $\infty + 1$ in 1— 1 correspondence with each other. We shall not comment on this further as it is beyond the scope of this report. In analysis, all results for $n = \infty$ are arrived at by taking $n \longrightarrow \infty$, and by proving theorems for $n \xrightarrow{Lt} \infty$, on the basis of finite values of n only, so that the paradoxes pertaining to ∞ are avoided {RUS 1}.

We shall next consider the paradoxes connected with the names of Russell and Richard, both of which also have in common the feature of a statement referring to itself in the negative, which has been the basis for all later development of incompleteness theorems in logic associated with Gödel and others.

(v) Russell's Paradox

Russell's paradox is closely similar to Cantor's, but it is based on a set being in one class or another and the name of the class which consists of all members of this class. We give this in some detail below as it seems to be the starting point for Richard's paradox which later led to Gödel's incompleteness theorem.

The line of argument runs as follows: Given any class of things C for example, either C is a member of itself or it is not — one or the other, but not both. When a class is not a member of itself, we may call it an "exclusive" class. Consider now the class of exclusive classes — say K. The object is to determine whether K is itself exclusive or not.

It is quite easy to see that if K is exclusive, then it is not a member of itself so that K is not exclusive. Vice versa, if K is not exclusive, then K is a member of itself. Thus, the answer to the question "Is K exclusive or not?" turns out to be neither, because each leads to the negation of itself. Therefore, from our general solution to Eq.(3) either the statement is impossible (truth value = XX) or it can only be said that it may or may not be true, but cannot be definitely proved to be true or false (truth value B).

(vi) Richard's paradox[5]

Russell's paradox dealing with sets has been generalized in Richard's paradox to definitions in a language. We consider this because it is the starting point for Gödel's incompleteness theorem. Now all definitions in a language can be listed by some rule. Let n be the listing number of one such definition. Now we ask the question whether n has the property of being the listing number of the definition of n itself. We call it Richardian if n does not have the property demanded by the defining expression listed as n in the example.

Then, the paradox named after Richard is the answer to the question "Is n Richardian or not?". Obviously, n is Richardian if and only if it does not have the property of the definition in the list under n. This is precisely the type of paradox we have been dealing all along, namely $\neg R(n) \Longleftrightarrow R(n)$. Once again we see that the only answer is either that no R(n) with the above definition exists, or that it cannot be said definitely that it has the property of being either Richardian or not Richardian, corresponding to truth values XX and B for $\underset{\sim}{R}(n)$. There is no logical invalidity, as R(n) is a valid statement by the rules of logic, but need not be always true or false.

In this case, the definition of 'Richardian' was brought in by a complicated definition in a language, which may not be normally admissible, unless the state B is accepted. What Gödel attempted was to obtain a definition of this type which is stated in mathematics— number theory — by rules, definitions and derivations based on axioms of arithmetic and he was able to get a very similar paradox, but which he showed has the truth value T, by following a different path.

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[5] A good account of this paradox and its relationship to Gödel's proof is given in [MN-1].